In grade school, we were presented with the sequence of natural numbers, with the rational numbers, with countless rules of arithmetic and geometry, and with complicated algorithms. We took these objects, rules, and algorithms to be axiomatic. The numbers 1, 2, 3, and so on are the natural numbers, and that is what numbers are. $1 + 1 = 2$ just because it does. If you want to prove it, 1 apple plus 1 apple equals 2 apples, said our grade school teacher. *A* squared plus *B* squared equals *C* squared if and only if the triangle *ABC* is a right triangle, because that is simply how a right triangle works. And besides, Euclid said that it is so.

Yet, a system with ten thousand rules is not a logically sound system. It is like the eight hundred page calculus book with no proofs, just countless rules for the differentiation and integration of very specific cases. For a logically sound, beautiful system of arithmetic, we trace our rules back to the source and find a very small set of logical truths and primitive notions on which to found our whole science. Then we build the entire system up using nothing but those notions. If we are careful, the entire system now rests on a few very well selected assumptions. These undefined assumptions are our axioms. The truth or falsehood of every theorem which we have produced is determined solely by our initial axioms, plus our chosen laws of logic. The fact that we have initially chosen only a few, non-controversial postulates to start with means that our final conclusions, no matter how wild and unlikely, can only be attacked on the grounds of our carefully picked and non-objectionable preliminary statements. This is our axiomatic system. In a sense then, a*xioms* may be defined as *that which are undefined*. Less cryptically, when we trace a chain of proofs and definitions back to their source, we must eventually arrive at things which we cannot prove, and which we cannot define in any simpler terms, and must assume to be true. These are the axioms of our system.

Book I of the *Elements*.		
Postulate 1. To draw a straight line from any point to any point. **Postulate 2.** To produce a finite straight line continuously in a straight line. **Postulate 3.** To describe a circle with any center and radius. **Postulate 4.** That all right angles equal one another. **Postulate 5.** Parallel postulate.	45 Theorems ⇒	Pythagorean Theorem. $A^2 + B^2 = C^2$ if and only if ABC is a right triangle.

Euclid's *Elements* is the model for axiomatic systems. *Book I* of the *Elements* begins with twenty three definitions, five postulates, and five common notions. The five postulates are usually considered the "axioms", but the entire set of statements, plus a few unstated assumptions, is Euclid's axiomatic system. Following a wonderful chain of forty five theorems, Euclid arrives at the final two theorems of the book, which make the bold assertion that given *ABC*, a triangle, $A^2 + B^2 = C^2$ if and only if the triangle is a right triangle. In the intervening pages, Euclid proves important theorems that will form the basis of the rest of geometry. By the time we reach the seventh book, Euclid is making significant discoveries about the theory of numbers using only his geometry axioms, despite the lack of a modern notational system for numbers. In books seven through nine we find theorems such as the Euclidean algorithm, proof that there are an infinite number of primes, and propositions proving the existence of irrational numbers (Heath, 1956). Euclid's *Elements* makes the case for the beauty, efficacy, and durability of the axiomatic method.

Since Euclid's time, our understanding of postulational systems has evolved. Euclid, of course, believed his axioms to be true based upon the nature of the physical world. Today, we recognize that any axiomatic system that is consistent may have validity, and may have real world applications. Systems such as Euclid's are particularly nice, because they do describe large parts of the real world. Yet now we know that Euclid's axioms minus the parallel postulate describe very unusual, formerly controversial, and yet entirely consistent geometries which have since been shown to be applicable to physical space.

Three Geometries		
Geometry	**Given Postulates**	**Result**
Euclidian	**Postulate 5.** Through any point not on a line there exists exactly one line parallel to the given line.	Interior angles of any triangle always sum to two right angles.
Hyperbolic	**Postulate 5.** Through a point not on a given line there exists more than one line parallel to the given line.	Interior angles of any triangle sum to less than two right angles.
Spherical	**Postulate 1.** Any two points determine at least one straight line. **Postulate 2**. A line is unbounded. **Postulate 5.** Any two lines in a plane intersect.	Interior angles of any triangle sum to greater than two right angles.

The risk of not having a clear axiomatic framework is that an entire mathematical field may be resting on an unstable foundation. One logical crack in the foundation negates the truth of all the field. Gottlob Frege describes this risk:

> Most mathematicians rest content, in enquiries of this kind, when they have satisfied their immediate needs. If a definition shows itself tractable when used in proofs, if no contradictions are anywhere encountered, and if connections are revealed between matters apparently remote from one another, this leading to an advance in order or regularity, it is usual to regard the definition as sufficiently established, and few questions are asked as to its logical justification. ..... By these methods we shall, at bottom, never have reached more than an empirical certainty, and we must really face the possibility that we may still in the end encounter a contradiction which brings the whole edifice down in ruins. For this reason I have felt bound to go back rather further into the general logical foundations of our science than perhaps most mathematicians will consider necessary. (Frege, 1884)

These contradictions which Frege speaks of have occurred several times in the history of mathematics. For example, in the years following Cantor's invention of set theory, no clear rules existed as to the creation of sets. It took over 20 years for Bertrand Russell to discover the fatal flaw in Cantor's set theory. Now, we are all familiar with Russell's antinomy:

Let *R* be the set of all sets which are not members of themselves.

Then *R* is an element of *R* if and only if *R* is not an element of *R*.

The great irony here is that Russell's antinomy also applies to some of Frege's most important work (O'Connor, Robertson, 2002).

Another instance of foundational flaws is in the history of calculus. From the time of Newton and Leibniz, calculus was developed and expanded. Machines, buildings, and the modern world were built using calculus. But the whole structure was teetering on an abyss due to a hazy notion of number. Calculus, in fact, rested on the back of fuzzy, undefined concepts called *fluxions* and *infinitesimals*. This opened calculus up to attacks on it's lack of rigor. George Berkeley mercilessly hammers these flaws in a pamphlet called *The Analyst*, in 1734:

> ... And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities ... ? (Newman, 1956)

Though not a mathematician, Berkeley's criticisms were valid. Still, it was 100 more years until Cauchy set calculus on a firm footing (Grabiner, 1981). At last, Weierstrass gave us our modern definition of *limit*:

> Let f be a function defined on an open interval containing c (except possibly at c itself) and let L be a real number. The limit as x approaches c of $f(x) = L$ means that for each real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for all x with $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$.

Finally, in 1872, Dedekind to gave us his famous theorem, which places irrational numbers and calculus firmly within on the domain of *number*:

> If a is any definite number, then all the numbers of the system R fall into two classes, A_1 and A_2, each of which contains infinitely many individuals; The first class A_1 comprises all the numbers a_1 that are $< a$, the second class A_2 comprises all the numbers a_2 that are $> a$; the number a itself may be assigned at pleasure to the first or second class, being respectively the greatest number of the first class or the least of the second. In every case the separation of the system R into the two classes A_1, A_2 is such that every number of the first class A_1, is less than every number of the second class, A_2. (Dedekind, 1963)

These definitions, now standard fare, were the result of nearly 200 years of effort by some of the world's finest minds to place calculus on a firm axiomatic basis. Generations of calculus students memorize the definition of *limit* without realizing that this definition is the keystone which keeps calculus from falling down around itself. But the structure is complete: as long as we accept the existence of the natural numbers, we must accept the truth of calculus.

Assembling Arithmetic

There are several ways to construct our system of arithmetic axiomatically. One way uses Peano's postulates.

Peano's Postulates

Logical truths:

1. $x = x$ for every x.
2 If $x = y$ then $y = x$.
3. If $x = y$ and $y = z$ then $x = z$.

Axioms:

1. 1 is a natural number.
2. For each x there exists exactly one natural number, called the successor of x, which will be denoted by x'. Thus if $x = y$ then $x' = y'$.
3. We always have $x' \neq 1$.
4. If $x' = y'$ then $x = y$.
5. Let there be given a set $\boldsymbol{M}$ of natural numbers, with the following properties:
   I. 1 belongs to $\boldsymbol{M}$.
   II. If x belongs to $\boldsymbol{M}$ then so does x'.

Then $\boldsymbol{M}$ contains all the natural numbers.

It is fairly simple to construct our number system from these axioms. E. Landau (1951) completes this construction in *Foundations of Analysis*. He uses just these rules to create the natural numbers, the associative and commutative laws, the laws of addition and multiplication, the laws of ordering, the rational numbers, the irrational numbers, Dedekind cuts, and complex numbers. He considers this a worthy exercise for any student of calculus in order to prove to themselves that the numbers used in calculus exist on a logical basis.

It is interesting to compare Peano's axioms with a set of axioms which use standard abstract

Abstract Algebra Axioms

Define a system of natural numbers, with two operations, $\{N, +, *\}$.
Axiom A. Identity. N contains an element such that $a * 1 = 1 * a = a$
Axiom B. Distributive. $a * (b + 1) = (a * b) + a$
Axiom D. Associative. $a + (b + 1) = (a + b) + 1$
Axiom E. Trichotomy. For any $a, b, \in N$, exactly one holds:

$a = b$
$a + x = b$
$a = b + y$

Axiom F. Induction. If M is a subset of N such that $1 \in M$ and $(k + 1) \in M$ when ever k is in M, then M is equal to N (Deskins, 1964).

algebra definitions. Peano's axioms are much more primitive, but the abstract algebra axioms allow us to begin on more familiar ground. Either way, we end up with the same system of arithmetic.

The actual construction of arithmetic using these axioms from abstract algebra is similar to that using Peano's postulates. Alternately, we might construct arithmetic using set theory. Using sets, we do not even need to assume the existence of the natural numbers. Instead, we need the concept that two sets are *similar* when the members may be put into one-to-one correspondence with each other. From Russell, we have the definitions:

> The *number of a set* is the set of all those sets that are similar to it.
> A *number* is anything which is the number of some set. (Russell, 1920)

To create the natural numbers from the Zermelo-Fraenkel axioms, we require three of the common ZF axioms:

The axiom of the null set.
There exists a set with no elements.
$\exists x \forall y \neg (y \in x)$
There exists *x* such that for all *y* it is not the case that *y* is an element of *x*.
The axiom of the pair set.
For any sets *x*, *y*, there is a set *z* that contains only *x* and *y* as members.
$\forall x \forall y \exists z \forall w\ (w \in z \leftrightarrow w = x \cup w = y)$
For all *x*, *y*, there exists *z* such that for all *w*, *w* is an element of *z* if and only if *w* = *x* or *w* = *y*.
The axiom of extensionality.
Two sets are identical when they have the same elements.
$\forall x \forall y \forall z\ ((z \in x \leftrightarrow z \in y) \rightarrow x = y)$
For all *x*, *y* , *z*, if *z* is an element of *x* if and only if *z* is an element of *y*, then *x* is equal to *y*.

Using our ZF axioms, we now construct the natural numbers. First, our axioms assure us that there exists at least one set, the *null set*, or empty set, ∅. This set has no members. Suppose *x* = ∅ and *y* = ∅. Then {∅, ∅} is a set, by the axiom of the pair set. We replace the set {∅, ∅}by the set {∅}, because they have the same members and are identical by the axiom of extensionality. The set of the null set has one member. Suppose that *x* = ∅ and *y* = {∅}. Then, by the axiom of the pair set again, the set {∅, {∅}}exists. This set contains two elements. We continue.

Set	Number
∅	0
{∅}	1
{∅, {∅}}	2
{∅, {∅}, {∅, {∅}}}	3
{∅,{∅},{∅, {∅}},{∅, {∅}, {∅, {∅}}}}	4

The natural number "0" represents any set which may be put into one-to-one correspondence with the null set. The natural number "1" represents any set which can be put into a one-to-one correspondence with the set of the null set. The number "2" represents any set which can be put into a one-to-one correspondence with the set that contains the null set and the set of the null set. In essence, using set theory, we create numbers out of nothing. One of the great realizations here is that the concept of one-to-one correspondence is a more primitive concept than that of number. One can recognize when two collections contain the same number of members, without knowing how many members that there are. The often used analogy is this: a Shepard puts one small pebble in his pocket as each sheep goes out to pasture in the morning. In the evening, he measures out one pebble as each sheep returns. If there are pebbles remaining after the sheep return, he knows that sheep have disappeared, without even knowing how to count.

Generating Geometry

Geometry may also be built up using axioms different from Euclid's. David Hilbert does so in *Foundations of Geometry*. Here, Hilbert seeks to correct flaws in the *Elements* (flaws which it took over 2000 years to find and correct). For instance, the very first proposition of the very first book of the *Elements* opens with the assumption that two circles drawn side by side will intercept at some point. Yet nothing in Euclid's postulates guarantees that this is true. (Eves, 1963). What is needed in this case are some kind of axioms of the continuity of lines. Hilbert provides these needed postulates, by dividing his axioms into five groups: axioms of connection, axioms of order, axiom of parallels, axioms of congruence, and axiom of continuity. The resulting set of twenty one axioms, although much larger than Euclid's, is also more thorough. We have such axioms as:

> Of any three points situated on a straight line, there is always one and only one which lies between the other two.

In the Euclidean system, this axiom would have been considered obvious based upon a drawing of a line. But in a modern system, we seek to reason not through drawings. Similarly, we have:

> Let A, B, C be three points not lying on the same straight line and let *a* be a straight line lying on the plane ABC and not passing through any of the points A, B, C. Then if the straight line *a* passes through a point of the segment AB, it will also pass through either a point of the segment BC or through a point of the segment AC.

This axiom states that if a line enters a triangle, it must exit the triangle through another side. Once again, this fact seems obvious from a drawing. But in a strict treatment, it needs to be an axiom. In a similar way, Hilbert patches a variety of foundational flaws found in Euclid.

Finally, in 1932, Birkhoff manages to reduce Euclidean geometry to four simple postulates, based upon the real number system (Peil, 2007).

Postulate of Line Measure. The points of any line can be placed in one to one correspondence with real numbers.
Point-Line Postulate. One and only one line contains two given points.
Postulate of Angle Measure. Shows how to measure angles.
Similarity Postulate. Shows when triangles are similar. .

Propping Probability

Amazingly, even a branch of mathematics as such as probability, which is seemingly based on inductive reasoning rather than the strict deductive reasoning of pure mathematics, may be axiomatized, and subjected to logic. Kolmogorov did just this, in 1932.

Axiom 1. $\mathbb{F}$ is a field of sets.
Axiom 2. $\mathbb{F}$ contains the set *E*.
Axiom 3. To each set *A* in $\mathbb{F}$ is assigned a non-negative real number P(*A*). This number is called the probability of the event *A*.
Axiom 4. P(*E*) equals 1.
Axiom 5. If *A* and *B* have no element in common, then P(*A*+*B*) = P(*A*) + P(*B*) (Kolmogorov, 1956, p. 5).
Axiom 6. Axiom of continuity (concerns only infinite fields).

A little reflection shows that these postulates reflect the things we know about probability. For instance the probability of event *A*, plus the probability of *not* the event *A*, equals 1.
$P(A) + P(\neg A) = 1$
Proof
$P(A) + P(\neg A) = P(A+\neg A)$ Axiom 5. ($A+\neg A$ have no element in common).
$P(A+\neg A) = P(E)$ Either *A* or not *A*.
$P(A) + P(\neg A) = P(E) = 1$ Axiom 4.
We may continue, and establish all of the usual theorems for addition and multiplication of probabilities.

The search for the foundations of mathematics has resulted in the unification of logic and mathematics, the birth of new branches of mathematics such as set theory, the genesis of new forms of geometry, and an evolution in the philosophy of mathematics. What the axiomatization of mathematics has not resulted in, and never will result in, is the creation of one perfect, logically sound, consistent, complete axiomatic basis for all of mathematics. In a way similar to that which the ancients were stymied by Euclid's parallel postulate, we are today thwarted by postulates such as the axiom of choice. Similarly to the way that the parallel postulate is independent of and consistent with the other postulates of geometry, the axiom of choice is independent of and consistent with the other postulates of Zermelo-Fraenkel set theory. In non-mathematical terms, the axiom of choice states:

> Let *C* be a collection of nonempty sets. Then we can *choose* a member from each set in that collection. In other words, there exists a function *f* defined on *C* with the property that, for each set *S* in the collection, *f*(*S*) is a member of *S* (Schechter, 2008).

This definition sounds innocuous, until we realize that this axiom implies puzzling functions such as, given the set of all the subsets of the real numbers, the choice function which chooses one member of each subset. Then we find that the axiom of choice also leads to such wonderful things as the Banach-Tarski paradox: that a sphere may be dissembled and reassembled into two identical copies of the original sphere. Yet we need the axiom of choice. Such innocent and basic statements as e*very vector space has a basis* may be shown to be equivalent to the axiom. Such plausible and obvious statements as *the union of countably many countable sets is countable* are absolutely dependent on the axiom of choice for their proof (Lopez-Ortiz, 1998) (countable here means that the sets may be put into a one-to-one correspondence with the natural numbers).

In 1932, Kurt Gödel proved that all of the inconsistencies can never be removed from mathematics. In essence, he created an algorithm for creating un-provable statements that works in any in any axiomatic system. Based upon Gödel's proof, and other contradictions above, we might even say that investigation of the foundations of mathematics has resulted in a lost innocence. Hence, it is hard today to maintain the belief that all of mathematics is built on some Platonic ideal of the concept of "number." In light of the discovery that slightly different forms of the rules of logic lead to different mathematics, it is difficult to hold even a purely logicist view of mathematics. Looking at in another way, however, we might look at these fundamental cracks in our axioms to be merely a deepening of the mystery of mathematics.

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