Euclid was the first non-Euclidean geometer. −H. S. M. Coxeter

Euclid states his fifth postulate as:

> *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side which are the angles less than two right angles* (Heath, 1956, p.202).

The fifth postulate has been stated in numerous other equivalent forms, most of which were drafted as a result of the labors to reduce the fifth postulate to the other four. Heath lists no less than sixteen of these, ranging from the common sense "*If a straight line intersects one of two parallels, it will intersect the other also*," due to Proclus in the fifth century, to the counter-intuitive "*There exists no triangle in which every angle is as small as we please*." due to Worpitzky in the nineteenth century. Frequently, these equivalent assumptions are the result of attempts at a proof of the fifth postulate. John Wallis, for instance, assumes that "*given a figure, another figure is possible which is similar to the given one and of any size whatsoever.*" In the end, all of these assumptions are exactly equivalent to the parallel postulate (Heath, 1956). One may even prove that the Pythagorean Theorem is equivalent to the parallel postulate (Bogomolny, 2008).

One reason why mathematicians have worked so hard to prove the parallel postulate is that, based upon how lines behave on a piece of paper, it seems inconceivable that parallel lines might intersect, or that skew lines might fail to intersect. The parallel postulate, it would appear, *must* be the consequence of the other, simpler, axioms of geometry. One plausible way to prove this is to assume that the 5th postulate is false, then show that a contradiction results. Eventually, repeated attempts of this nature led to the realization that *no* contradiction results. Further exploration resulted in the first tentative forays into non-Euclidean geometry, and to the dawning of the modern concept of mathematics, where mathematical truths are logical conclusions drawn from postulate sets, rather than statements which necessarily have a relation to the empirical world.

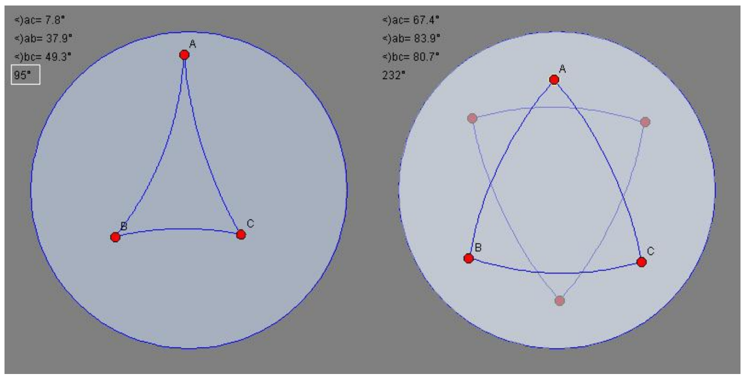
The first class of mathematicians who discovered theorems from non-Euclidean geometry are those who did not recognize the nature of their findings, but instead viewed them as contradictions arising from denial of the fifth postulate. First, we have Saccheri, who around the year 1700, generated two lines which have a common perpendicular at infinity. Then, in the mid 1700's, Lambert arrived at triangles with angles and sides which remain constant under transformation (Coxeter, 1942). Compare these triangles to the triangles of Euclidean geometry, where the angles remain constant under size transformations, but the sides change. Mathematicians of the time saw these contradictions as support for the view that the fifth postulate is a consequence of the nature of space: we see these phenomenon as defining characteristics of specific non-Euclidean geometries.

The second class of mathematicians who explored the nature of non-Euclidean geometry are those who kept their findings secret because they did not deem that the world was ready to accept them (perhaps not being interested in a fate similar to that of Galileo or Copernicus). The sole member of this class is Gauss. His attempts to prove the fifth postulate during his younger years finally led him to the modern belief that Euclidean geometry may be viewed as the geometry on the surface of an infinitely large sphere (Coxeter, 1942).

The third class of mathematicians who developed the science of non-Euclidean geometry are those whose work went largely unread and unacknowledged until they were other than living. In this class, we have Lobachevsky and Bolyai, who each independently explored and published work on hyperbolic geometry in the 1820's. Lobachevsky published his work in 1829, but it was not until more than 10 years after his death in 1856 that his work was widely read or distributed. Bolyai, as a young man, developed hyperbolic geometry as the result of a failed attempt to prove the parallel postulate. His results were published in 1831, and were even read by Gauss, yet failed to make any impact within his lifetime (O'Connor and Robertson, 2000).

The forth and final class of mathematicians who explored non-Euclidean geometry are those who released their findings to a mathematical culture that was ready to understand and accept them. Here, we have the mathematicians who finally proved that the parallel postulate is independent of the other four postulates of Euclidean geometry, such as Beltrami, Cayley, Klein, and Poincare. We also have Riemann, who discovered many new types of geometry (Eves, 1963), and who in 1854 speculated as to the relation between non-Euclidean geometry and physical space (O'Connor and Robertson, 2000), which would profoundly influence Einstein, and all thought after the dawn of the 20^{th} century.

There are numerous reasons why an understanding of non-Euclidean geometry is valuable in the teaching of high school mathematics. Geometry is the subject where students first encounter the concept of *proof*, and of the axiomatic method. The differences between Euclidean, hyperbolic, and spherical geometry are a vivid way to illustrate the variations that arise, simply by altering a few postulates. Then, a comparison of theorems that hold in only one form of geometry, or in several, may be easily demonstrated. A further exploration of these alternative geometries via the computer makes these differences more visual, more tactile, and more exploratory. A presentation of Euclidean geometry which acknowledges other geometries places Euclidean geometry into context without diminishing its importance, while also providing a more modern view of mathematics, and of actual physical space.



Finite geometries are another form of non-Euclidean geometries which are a highly effective way to introduce students to abstract thinking and creativity within mathematics. The stripped down nature of finite geometry postulate sets make them accessible to anyone. Take for instance this very basic example:

1. There exist exactly three points.
2. Not all points are on the same line.
3. On any two distinct points there is exactly one line.
4. On any two distinct lines there is at least one common point.

The resulting figure is a triangle. Still, there are a wealth of provable theorems, and the undefined terms may be interchanged, or replaced with other words, as a valuable illustration of what is meant by the abstract nature of mathematics (Schaaf, W. L. 1969). Again, finite geometries do not diminish the importance of Euclidean geometry, they emphasize the significance of the axioms which Euclid chose. At one time, the thought that axioms are arbitrarily chosen would have been revolutionary. At the dawn of the 21st century, we have had several generations to become accustomed to such things as arbitrary computer languages. These thoughts lead naturally to such discussions as, if arithmetic is also based upon axioms, and those axioms are completely arbitrary, is our system of arithmetic simply something arbitrary?

Finally, we have a word from Howard Eves:

In fact, it is no exaggeration to say that the historical consideration of the importance of Euclid's parallel postulate is responsible for initiating the entire study of properties of postulate sets and hence for shaping much of the modern axiomatic method (Eves, 1963, p.398).

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