The statement "**The *n* by *n* matrix *A* is invertible**" may be restated in many equivalent ways. Many of these restatements reveal different properties of matrices, such as the null space, the column space, the row space, the solutions, the determinant, and more.

A definition of invertibility provides our first restatement. **For the *n* by *n* matrix *A*, there exists another matrix (A^{-1}) such that $AA^{-1} = I$, and $A^{-1}A = I$, where *I* is the *n* by *n* identity matrix**. The advantage of this definition is that it implies the existence of another matrix, the inverse of *A*. Given that *A* is invertible, we know that this inverse exists. Given the inverse, we know that the matrix *A* is invertible. The above definition also makes it clear that the left inverse and right inverse are identical. This fact is important in an area of mathematics where the commutative law seldom holds. Strictly speaking, the "*n* by *n*" part of our definitions are not needed, because only square matrices may have both left and right inverses. We may thus merely state, "**The matrix *A* is invertible.**" Alternatively, we may state **"The matrix *A* is non-singular,"** because we define a singular matrix to be one which is not invertible, and any matrix is either invertible or not-invertible, singular or non-singular.

The next set of restatements which we encounter are those which concern elementary matrices and row-operations. The fundamental characterization here is "***A* is a product of elementary matrices**." To explain this statement, we must first define an *n* by *n elementary matrix* as being one that can be obtained from the *n* by *n* identity matrix with only one *elementary operation*. We define an elementary operation as being (i) multiplying the one row of the identity matrix by a non-zero scalar, (ii) multiplying one row of the identity matrix by a non-zero scalar then adding it to another row, or (iii) swapping two rows of the identity matrix. There is exactly one elementary matrix corresponding to each elementary operation for an *n* by *n* matrix. The matrix corresponding to a specific elementary operation is the matrix which performs that same operation when it multiplies another matrix.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Do nothing.

$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ Multiply first row by -2 and add it to second row.

$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ Multiply first row by 3.

Some elementary matrices and operations.

Each elementary matrix has an inverse, which performs the inverse elementary operation. All matrices which may derived from, or reduced to, the identity matrix are products of elementary matrices. For an invertible matrix *A*, there must be some A^{-1} such that the product of *A* and A^{-1} is the identity matrix. Hence A^{-1} is equal to the product of a set of matrices which when multiplied by *A* yield the identity matrix. Also *A* is equal to the product of the inverse of this same set of matrices, which when multiplied by the matrix A^{-1}, yield the identity.

A numerical example is here illustrative. Let A be the 2 by 2 invertible matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then A^{-1} is the matrix $\begin{bmatrix} -2 & 1 \\ 1.5 & -.5 \end{bmatrix}$, because $A\,A^{-1} = I$: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} -2 & 1 \\ 1.5 & -.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Further, we notice that A^{-1} is equal to the product of three elementary matrices:

$$\begin{bmatrix}1 & -2\\0 & 1\end{bmatrix} * \begin{bmatrix}1 & 0\\0 & -0.5\end{bmatrix} * \begin{bmatrix}1 & 0\\-3 & 1\end{bmatrix} = \begin{bmatrix}-2 & 1\\1.5 & -.5\end{bmatrix} = A^{-1} \qquad E_1E_2E_3 = A^{-1}$$

The inverse of this product is A:

$$\left(\begin{bmatrix}1 & -2\\0 & 1\end{bmatrix} * \begin{bmatrix}1 & 0\\0 & -0.5\end{bmatrix} * \begin{bmatrix}1 & 0\\-3 & 1\end{bmatrix}\right)^{-1} \qquad (E_1E_2E_3)^{-1} = \mathrm{A}$$

$$= \begin{bmatrix}1 & 0\\-3 & 1\end{bmatrix}^{-1} \begin{bmatrix}1 & 0\\0 & -0.5\end{bmatrix}^{-1} \begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}^{-1} \qquad E_3^{-1}\, E_2^{-1}\, E_1^{-1} = \mathrm{A}$$

$$= \begin{bmatrix}1 & 0\\3 & 1\end{bmatrix} * \begin{bmatrix}1 & 0\\0 & -2\end{bmatrix} * \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = A.$$

An examination of our numerical example reveals that our elementary matrices correspond to the elimination steps in reducing our matrix to reduced row echelon form.

$$\begin{bmatrix}1 & 0\\-3 & 1\end{bmatrix} * \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & -2\end{bmatrix}$$ Subtract 3 times row 1 from row 2

$$\begin{bmatrix}1 & 0\\0 & -0.5\end{bmatrix} * \begin{bmatrix}1 & 2\\0 & -2\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$$ Multiply row 2 by -0.5

$$\begin{bmatrix}1 & -2\\0 & 1\end{bmatrix} * \begin{bmatrix}1 & -2\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$ Subtract 2 times row 2 from row 1.

The above discussion leads us to our next restatements: "***A* row reduces to the *n* by *n* identity matrix**." , "***A* is row equivalent to the *n* by *n* identity matrix**." and "**The reduced row echelon form of *A* is *I***."

The next class of equivalent statements describing invertibility which we tackle are those involving columns. We start with "**The *n* by *n* matrix *A* has *n* independent columns**." Independent columns are vectors which are not linear combinations of each other. We can instantly see that an invertible matrix has *n* independent columns, because, as we illustrated above, every invertible matrix may be row reduced to the identity matrix. The identity matrix has independent columns, as no linear combination of them can produce the zero vector, except for the zero relationship.

$$\alpha_1\begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha_2\begin{bmatrix}0\\1\\0\end{bmatrix} + \alpha_3\begin{bmatrix}0\\0\\1\end{bmatrix}$$ cannot equal zero unless α_1, α_2, and α_3 are zero.

The identity matrix has independent columns.

However, we do not need to appeal to the properties of the identity matrix to explore the properties of independent columns, but can develop these things based entirely on the characteristics of vectors. We first need a field, ***F***. The field can be any field such as the real numbers, the rational numbers, or the complex numbers. We call the elements of our field *scalars*. We then define *vectors* as being ordered sets of scalars. Next, we define the addition of two vectors with the same number of elements as being

$$(a_1, a_2, \dots , a_n) + (b_1, b_2, \dots , b_n) = (a_1 + b_1, a_2 + b_2, \dots , a_n + b_n).$$

We define the multiplication of a vector by a scalar as being

$$\alpha\ (a_1, a_2, \dots, a_n) = (\alpha\ a_1, \alpha\ a_2, \dots\dots, \alpha\ a_n)$$

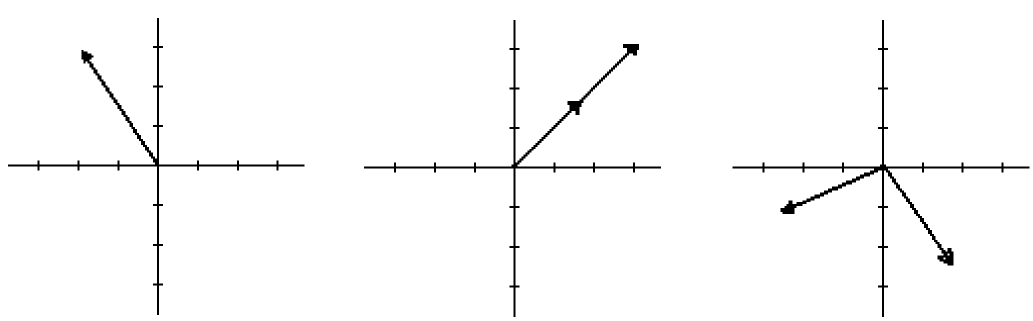
We say that vectors $X_1, X_2, \dots X_n$ are *linearly dependent* when there exist scalars, not all of which are zero, such that

$$\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_n X_n = 0.$$

We say that $X_1, X_2, \dots X_n$ are *linearly independent* if no such scalars exist, meaning that if $\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_n X_n = 0$, then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

With this much information about vectors, it simple to determine many crucial facts concerning the columns of matrices. We can conclude that the linear combinations of at least *n* independent vectors with *n* elements are required to produce every possible vector with *n* elements. We call this spanning the space $\boldsymbol{F}^n$. We can then see that less than *n* vectors (or columns) can never span $\boldsymbol{F}^n$. We can also prove that if we have more than *n* vectors with *n* elements, some of those vectors must be dependent. If, further, we define a *basis* as being a set of vectors which spans a space, we can conclude that every basis of $\boldsymbol{F}^n$ is a set of *n* linearly independent vectors, and that every set of *n* independent vectors is a basis for $\boldsymbol{F}^n$.

Geometrically, in R^n, we can see these things very easily. If we are in R^2, a plane, it takes the linear combinations of two independent vectors to fill the space of R^2. One vector cannot fill the space, and given 2 independent vectors, a third vector will always be dependent on the other two. In R^3, 2 vectors can never form more than a plane. 3 vectors may form a line, or a plane, or the entire 3 dimensional space. 4 vectors in R^3can result in at most 3 dimensions, so at least one of the vectors must be dependent.



One vector makes a line in R^2 — Two dependent vectors fail to span R^2 — Two independent vectors span R^2

With our discussion of vectors concluded, we can see the equivalence of our remaining statements concerning columns.

The vectors $v_1, v_2, ..., v_n$ are a basis for R^n.
The matrix A has n columns which span R^n.
The columns of A are a linearly independent set.
The n columns of A are a basis for R^n.
The column space of the n by n matrix A is R^n.

The above discussions of columns, identity matrices, and invertibility lead naturally to a description of rank and pivot columns. The *rank* of a matrix is equal to the number of independent columns in a matrix. Geometrically, the rank is equal to the number of dimensions that a set of columns spans. The rank of a matrix is not changed by elimination, and is of course equal to the number of leading non-zeros in an echelon matrix. We call the leading non-zeros *pivots*, and the columns they are in *pivot columns*. With these definitions in mind, our two statements involving ranks and pivots are self explanatory. We have:

The rank of the n by n matrix A is n.
The n by n matrix A has n pivot columns.

Proof: Invertible matrices row-reduce to the identity matrix. The identity matrix has one entry in each column. Hence the n by n identity matrix has rank n and has n pivot columns. Further, any n by n matrix which has n pivot columns or rank n must be row-reducible to the identity matrix. Elimination will produce zeros above and below each pivot, then the rows containing any pivots not equal to 1 can be multiplied by a scalar to produce a 1 in the pivot.

The next set of statements equivalent to "The n by n matrix A is invertible" which we will address are those involving the nullspace of the matrix. We start with "**The equation $Ax = 0$ has only the trivial solution.**" The *trivial solution* here means all zeros. We might say that the only x's that we may multiply by the columns of A to produce the zero vector are all zeros. But, we know this already, based upon our definition of independent vectors. The columns of A are independent exactly when $\alpha_1 X_1 + \alpha_2 X_2 + ... + \alpha_n X_n = 0$ implies $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$. Each set of $(\alpha_1, \alpha_2, ..., \alpha_n)$ that multiply the columns of A to produce zero are the solutions to $Ax = 0$. The columns of an invertible matrix are independent, so **$Ax = 0$ has no solution except for the zero vector.** We are justified in calling the vectors that multiply a matrix to produce the zero vector a *space*, since zero is always one of these vectors, and any linear combination of these vectors will also produce zero, so we might say **The nullspace of A contains only the zero vector.** Sometimes, we define the *nullity* of a matrix as being the number of columns minus the rank. Since the rank of an invertible matrix is equal to the number of columns, we could then say, **The nullity of A is zero.**

We can prove all of the above statements using matrix notation.

Assume A is invertible.

$Ax = 0$	For any A, some x satisfies this equation. (zero always will)
$A^{-1}Ax = A^{-1}0$	multiply both sides by A^{-1}
$(A^{-1}A)x = 0$	associative law
$Ix = 0$	$A^{-1}A$ = the identity
$x = 0$	

Therefore, for an invertible A, any x that satisfies $Ax = 0$ is equal to the zero vector.

The next group of statements we encounter are those that concern the solutions to the equation $Ax = b$. First, we have "**The equation $Ax = b$ has at least one solution for each b in R^n.**" The logic to this statement is: We assume the n by n matrix A is invertible. Thus A spans the space R^n. Hence all vectors with n components are in the column space of A. Hence there exists some linear combination of the columns of A that produces any vector in R^n. For a given b, the solution vector x is exactly this linear combination of the columns of A. Thus the equation $Ax = b$ has at least one solution for each b in R^n.

Symbolically, we can approach the problem like this:

Assume A is invertible

$Ax = b$

$A^{-1}Ax = A^{-1}b$ multiply by A^{-1}

$Ix = A^{-1}b$ $A^{-1}A = I$

$x = A^{-1}b$

Therefore **the only solution to $Ax = b$ is $x = A^{-1}b$.** We could rephrase this as **"The equation $Ax = b$ has a unique solution x for every b in R^n."** Further, we would be justified in stating **"The equation $Ax = b$ never has two solutions for any b in R^n."**

Next, we have "**The determinant of A is not zero**." We show that an invertible matrix does not have a determinant of zero, and that a matrix with a determinant of zero is not invertible.

Proof: Suppose that a matrix A is invertible. Then A is a product of elementary matrices. The determinant of the product of elementary matrices is equal to the product of the determinants of the elementary matrices. The determinants of elementary matrices are not zero. Therefore, when a matrix is invertible, its determinant is not zero.

Suppose that a matrix A, with a determinant of zero, is invertible. Since A is invertible, it has an inverse, and the determinant of A inverse is 1 divided by the determinant of A. But the determinant of A is zero, and 1 / 0 is undefined. Hence the determinant of A inverse does not exist. Therefore, when a matrix has a determinant of zero, it is not invertible.

As a final example, consider the equation $Ax = b$. For some square matrices, for some non-zero values of x, b is a scalar multiple of x. Then we write $Ax = \lambda x$. The vector x is an eigenvector of A and the scalar λ is an eigenvalue of A. To show that we can reasonably expect such things to happen , we produce the identity matrix. When any vector is multiplied by the identity matrix, the result is 1 times the original vector. So, any vector is an eigenvector of the identity matrix, with an eigenvalue of 1.

We arrive at our final statement equivalent to the invertibility of a matrix. **Zero is not an eigenvalue of A.**

Proof: For a non-invertible matrix, we always have non-zero x's that produce the zero vector. These are the vectors in our null space. They may also be considered as eigenvectors which each have an eigenvalue of zero. Thus, for an non-invertible matrix, zero is always an eigenvalue.

We have shown above that for an invertible matrix A, there is no non-zero vector x such that $Ax = 0$. Hence, when $x \neq 0$, it impossible to have $Ax = 0x$. So, for an invertible matrix, zero is never an eigenvalue.

We see above how statements concerning the property of invertibility reveal different aspects of what a matrix is, such as a set of linear equations with a certain number of variables, or a set of vectors in *n* dimensional space. Other statements illustrate properties of matrices, such as solutions, factorizations, determinants, and eigenvalues. All of the statements show how central the idea of invertability is to the subject of algebra. difonzo