The four great Greek problems, squaring the circle, trisecting the angle, doubling the cube, and constructing the *n*-gon, have occupied geometers and mathematicians for over 4000 years. The struggle to solve these problems, and the subsequent proof of the impossibility of solutions, has been one of the primary motivations for the development of mathematics. The story begins with the Egyptians, continues with the Greeks, and does not see any real solutions until Gauss's solution of the polygon problem in the year 1796 (Courant and Robbins, 1941). The saga finally ends in 1882 with Lindemann's proof of the transcendence of π (Beckmann, 1971). In the end, it is not geometry that solves the problems, but algebra and analysis.

The problems can be stated thusly: using only the 5 postulates of Euclid, how may we divide an arbitrary angle into three equal angles, how may we construct a square with the same area as a given circle, how may we construct a cube with twice the volume as an arbitrary cube, and how may we construct a polygon with a given number of sides? In practice, Euclidian constructions are those which can be made using only a straight edge and compass, so the problems become that of using only these two tools for these tasks.

The first of our problems to be mentioned is the problem of squaring the circle, which is found in the Rhind Papyrus dating from approximately 1650 B.C. Here, we are instructed that squaring 8/9th of the diameter of a circle will create a square with equal area to the circle (Klein, 1895). This operation actually results in a value for π of approximately $(2 * 8/9)^2 = 3.16$. We may consider this as the first of many thousands of erroneous constructions of the squared circle. Of course, the Egyptians were attacking the problem in a primarily practical way.

By the time of the Greeks, all of the problems were known and were being pursued from a theoretical basis. Although the Athenians' and Cyzicians' study of the problems from 420 B. C. to 300 B. C. does not lead to any solutions, they do advance geometry to a high degree. Plato, who lived during this era, is known for having heavily promoted the axiomatic method, and for having promoted the use of straight edge and compass alone (Ball, 1912).

Throughout the Greek era, attempts to solve our problems result in a wealth of ingenious new curves and constructions. For example, in approximately 420 B. C., Hippias of Elis gives the quadratrix, a curve which trisects the angle (Ball, 1912). Later, Archimedes (287-212 B.C.), gives a construction of the trisection of the angle in Proposition 8 of his Book of Lemmas (Heath, p. 564). In around 180 B. C., Nicomedes invents the conchoid, a curve which squares the circle (Smith, 1958). Many other works of the time give similar constructions. The common thread to all of these solutions is that they all require tools beyond the straight edge and compass, or involve making marks on the straight edge.

The first progress towards a modern solution to the Greek problems comes in the late 1500's, with the rise of algebra. Francois Viète is the first to recognize that the problem of trisecting the angle results in a cubic equation (Boyer, 1968). A century later, in 1667, James Gregory, who believed that "the circle-squarers were pursuing a vainer phantom than those who endeavor with rule and compass to trisect an angle (Turnbull, p. 139)" gives the first proof of the transcendence of π . Unfortunately, the proof proves to be incorrect.

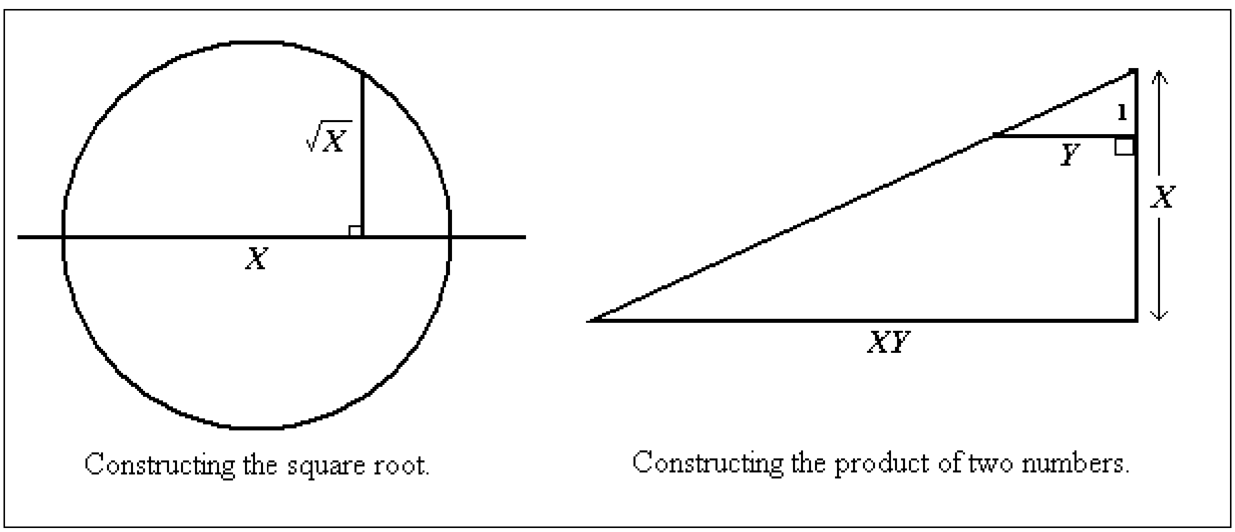
Finally, the rush towards solutions begins in 1796, when a young Gauss discovers that a 17 sided polygon is constructible, and determines that the only constructible polygons are those with a number of sides equal to a power of 2, a prime of the form $2^{2^n}+1$ (or 3, 5, 17, 257, 65,537 ...) or a multiple of these number of sides (Courant and Robbins, 1941). The list of constructible polygons is then those with 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, ... sides.

Curiously, a polygon with *n* sides is constructible if and only if $\varphi(n)$, the number of natural numbers less than and relatively prime to *n*, is a power of 2. No other polygons are constructible by ruler and compass. Using Euclidian tools, we won't ever arrive at a polygon with 7, 9, or 11 sides, for instance.

Gauss may or may not have had a proof of his findings. But very soon, in the early 1800's, Ruffini, Abel and Galios perfect the required mathematics. This is the birth of the modern theory of equations, abstract algebra, and group theory (Courant and Robbins, 1941). Finally, in 1837, Wantzel publishes a paper that proves the impossibility of the trisection an arbitrary angle. squaring a circle, and the doubling the cube, using the ideas of Abel and Galios (O'Connor & Robertson, 1997).

The first important concept in understanding the solution to our problems is that of closure. A simple example is the closure of the integers under the operation of addition. No matter what integers we may add or subtract from each other, we will never produce any other kind of number, other than integers We cannot ever arrive at ½, or ¾, or 0.003, or any other number that is not a positive or negative whole number. We will see that this concept of closure applies to more complex kinds of numbers, with more operations.

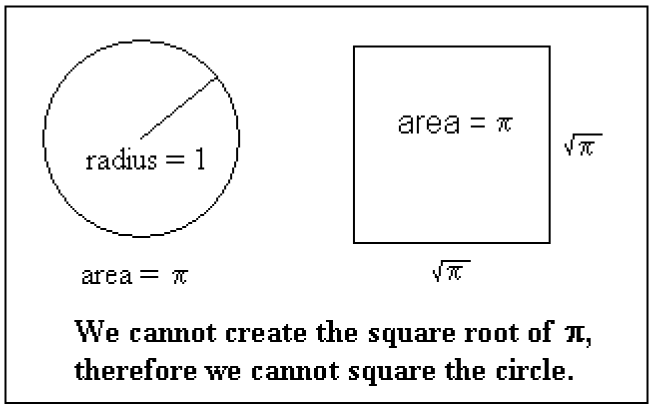
Returning to our Greek problems, we first note that using a straight edge and compass, we may perform only five operations. The first four are the operations of arithmetic. We may add two numbers together, we may subtract two numbers, we may multiply to numbers together, and we may divide two numbers. In geometry, a "number" is the magnitude of a line. So starting with the number 1, and performing these four operations, we may arrive at any rational number. We thus create the "field" of rational numbers.



Some Euclidian Constructions.

When we add to these operations the operation of taking the square root, we arrive at another bigger field. This field is the field of Euclidian constructions. We may perform arithmetic operations on any member of this field, then take a square root of the result, then perform arithmetic operations on that result, then take more square roots. No matter what operations we do with a ruler and compass, we can never get any type of number except those generated by combinations of additions, subtractions, multiplications, divisions, and square roots. What kinds of numbers are these?

It turns out that these numbers are the roots of equations with variables of the degree 2^n. The operations of Euclidian geometry are closed under this field: any finite number of combinations of Euclidian operations can never produce a number outside of this field. If we can determine that a certain magnitude lies outside of this field, we know that it is not constructible with straight edge and compass. Take for instance, the number $\sqrt[3]{2}$. A finite number of arithmetical operations and square roots will never result in a cube root. The problem of doubling the cube may be expressed in these terms. For the volume of a cube of one unit, we have $1^3 = 1$. The problem of doubling the cube is thus $x^3 = 2$, or $x = \sqrt[3]{2}$. A compass and straight edge will never get us to this x.



Similarly, given that π is not the root of any equation, we instantly know that we can not construct the magnitude π, so we can't construct $\sqrt{\pi}$, so we can not square the circle.

Euclid said to Ptolemy, "There is no Royal Road to geometry." Modern Galois theory and field theory have solved the problem of the solutions to roots of polynomials. But there is no Royal Road to Galios theory. In outlining Wantzel's solution to the trisection problem, we will need to take as assumptions some of the major results of Galois theory, and gloss over some details, as we have done above.

To show that the general angle is not trisectible, we need only give one example of an angle which may not be trisected. Let this angle be $\frac{\pi}{3}$. We will show that we can not construct $\frac{\pi}{9}$.

We will use the identity $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$.

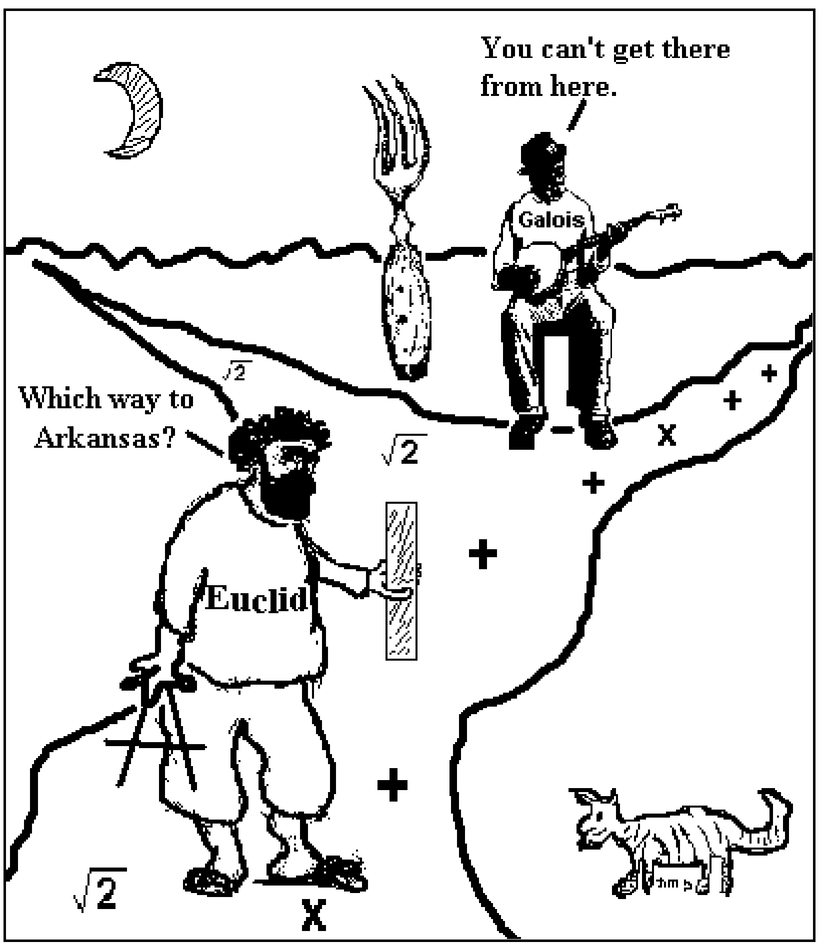
Let $\theta = \frac{\pi}{9}$.

Then $\cos(3\theta) = ½$.

Let $x = \cos(\theta) = \cos(\frac{\pi}{9})$.

Substituting in our identity, we get ½ = $4x^3$ -3x
or $8x^3-6x-1 = 0$
A theorem from the theory of equations states that if a cubic equation with rational coefficients has no rational root, than none of its roots is constructible (Bold, 1969). $8x^3-6x-1 = 0$ has no rational roots, therefore $\frac{\pi}{9}$ is not constructible, therefore $\frac{\pi}{3}$ is not trisectible (Rotman, 1996).

The wonderful thing about mathematics is that concepts that are not true in the real world may be quite true in mathematics. In the real world, "You can't get there from here" is a joke: you can get anywhere from anywhere. In mathematics "You can't get there from here" is a very real and very significant concept. Often, mathematical concepts turn out to be part of the real world too. For example, only a few centuries ago the idea of negative numbers was seen as radical, impossible, or nonsensical. Today, we all know about negative bank accounts. In Newton's time, the idea of instantaneous velocity was highly questionable. Now we observe the speedometer in our automobile without the tiniest bit of skepticism.



Who knows what the real world implications of proven impossibility are?

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